数理解析研究所講究録
1012巻 1997年 115-139

A CLASS OF FUNCTIONS DEFINED BY USING HADAMARD PRODUCT. II

S. Owa, H. M. Hossen and M. K. Aouf

ABSTRACT. The object of the present paper is to obtain closure theorems, integral operators and several interesting results for the modified Hadamard products of functions belonging to the class $P_\alpha[\beta,\gamma]$ consisting of analytic functions with negative coefficients and defined by using Hadamard product $f * S_\alpha(z)$ of $f(z)$ and $S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$. Also we obtain distortion theorem for certain fractional integral operator of functions in the class $P_\alpha[\beta,\gamma]$.

KEY WORDS- Analytic, starlike, modified Hadamard product.

AMS (1991) Subject Classification. 30C45.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. And let S denote the subclass of A consisting of analytic and univalent functions f(z) in the unit disc U. A function f(z) in S is said to be starlike of order α if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U) \tag{1.2}$$

for some $\alpha(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Further, a function f(z) in S is said to be convex of order α if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U) \tag{1.3}$$

for some $\alpha(0 \le \alpha < 1)$. And We donote by $K(\alpha)$ the class of all convex functions of order α. It is well-known that

$$f(z) \in K(\alpha) \text{ if and only if } zf'(z) \in S^*(\alpha). \tag{1.4}$$

These classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [8], and later were studied by Schild [9], MacGregor [3] and Pinchuk [7].

Now, the function

$$S_{\alpha}(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (1.5)$$

is the well-known extremal function for the class $S^*(\alpha)$ (see [8,1]). Setting

$$C(\alpha,n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!} \qquad (n \geq 2). \qquad (1.6)$$

$S_{\alpha}(z)$ can be written in the form

$$S_{\alpha}(z) = z + \sum_{n=2}^{\infty} C(\alpha,n)\, z^n. \qquad (1.7)$$

Then we note that $C(\alpha,n)$ is decreasing in α and satisfies

$$\lim_{n\to\infty} C(\alpha,n) = \begin{cases} \infty & (\alpha < \frac{1}{2}) \\ 1 & (\alpha = \frac{1}{2}) \\ 0 & (\alpha > \frac{1}{2}). \end{cases} \qquad (1.8)$$

Let $f * g(z)$ be the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n\, z^n, \qquad (1.9)$$

then

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.10}$$

We say that the function f(z) defined by (1.1) belongs to the class $P_\alpha(\beta,\gamma)$ if f(z) satisfies the following condition

$$\left| \frac{(f * S_\alpha(z))' - 1}{(f * S_\alpha(z))' + (1-2\beta)} \right| < \gamma \qquad (z \in U) \tag{1.11}$$

for $\beta(0 \leq \beta < 1)$ and $\gamma(0 < \gamma \leq 1)$.

Let T denote the subclass of A consisting of functions f(z) of the form

$$fz) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0). \tag{1.12}$$

And we denote by $P_\alpha[\beta,\gamma]$ the class obtained by taking intersection of $P_\alpha(\beta,\gamma)$ with T, that is,

$$P_\alpha[\beta,\gamma] = P_\alpha(\beta,\gamma) \cap T. \tag{1.13}$$

The class $P_\alpha[\beta,\gamma]$ was studied by Owa and Ahuja [6]. The class $P_\alpha[\beta,\gamma]$ is the generalization of the class $P^*(\beta,\gamma)$ which was introduced by Gupta and Jain [2]. In particular, $P_{1/2}[\beta,\gamma] = P^*(\beta,\gamma)$ when $\alpha = \frac{1}{2}$. Further we note that many classes defined by using the Hadamard product $f * S_\alpha(z)$ of f(z) and $S_\alpha(z)$ were introduced by Sheil-Small, Silverman and

Silvia [11], Silverman and Silvia ([12], [13]), and Ahuja and Silverman [1].

In order to prove our results for functions belonging to the class $P_\alpha[\beta,\gamma]$, we shall require the following lemma given by Owa and Ahuja [6].

Lemma 1. Let the function f(z) be defined by (1.12). Then f(z) is in the class $P_\alpha[\beta,\gamma]$ if and only if

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\ a_n \leq 2\gamma(1-\beta). \tag{1.14}$$

The result is sharp.

2. Closure Theorems

Let the functions $f_i(z)$ be defined, for $i = 1,2,\dots, m$, by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i}\ z^n \qquad (a_{n,i} \geq 0) \tag{2.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $P_\alpha[\beta,\gamma]$.

Theorem 1. Let the functions $f_i(z)$ $(i = 1, 2, \ldots, m)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{2.2}$$

also belongs to the class $P_\alpha[\beta,\gamma]$, where

$$b_n = \frac{1}{m} \sum_{i=1}^{m} a_{n,i}. \tag{2.3}$$

Proof. Since $f_i(z) \in P_\alpha[\beta,\gamma]$, it follows from Lemma 1 that

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i} \le 2\gamma(1-\beta) \quad (i=1,2,\ldots,m). \tag{2.4}$$

Therefore

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)b_n = \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\left[\frac{1}{m}\sum_{i=1}^{m} a_{n,i}\right]$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left\{\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i}\right\} \le 2\gamma(1-\beta). \tag{2.5}$$

Hence, by Lemma 1, $h(z) \in P_\alpha[\beta,\gamma]$. Thus we have the theorem.

Theorem 2. Let the functions $f_i(z)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ for each $i = 1, 2, \ldots, m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{m} d_i \, f_i(z) \qquad (d_i \geq 0) \tag{2.6}$$

is also in the same class $P_\alpha[\beta,\gamma]$, where

$$\sum_{i=1}^{m} d_i = 1. \tag{2.7}$$

Proof. According to the definition of $h(z)$, we can write that

$$h(z) = z - \sum_{n=2}^{\infty} \left[\sum_{i=1}^{m} d_i \, a_{n,i} \right] z^n. \tag{2.8}$$

By means of Lemma 1, we have

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i} \leq 2\gamma(1-\beta) \tag{2.9}$$

for every $i = 1, 2, \ldots, m$. Hence we can observe that

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n) \left[\sum_{i=1}^{m} d_i \, a_{n,i} \right]$$

$$= \sum_{i=1}^{m} d_i \left\{ \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i} \right\}$$

$$\leq \left[\sum_{i=1}^{m} d_i \right] 2\gamma(1-\beta) = 2\gamma(1-\beta) \qquad (2.10)$$

which implies that $h(z) \in P_\alpha[\beta,\gamma]$. Thus we have the theorem.

Theorem 3. The class $P_\alpha[\beta,\gamma]$ is closed under convex linear combination.

Proof. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$. Then it is sufficient to prove that the function

$$h(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \quad (0 \leq \lambda \leq 1) \qquad (2.11)$$

is in the calss $P_\alpha[\beta,\gamma]$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{n=2}^{\infty} \left\{\lambda a_{n,1} + (1-\lambda)a_{n,2}\right\} z^n, \qquad (2.12)$$

with the aid of Lemma 1, we have

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\left\{\lambda a_{n,1} + (1-\lambda)a_{n,2}\right\} \leq 2\gamma(1-\beta) \qquad (2.13)$$

which implies $h(z) \in P_\alpha[\beta,\gamma]$.

3. Integral Operators

Theorem 4. Let the function $f(z)$ defined by (1.12) be in the class $P_\alpha[\beta,\gamma]$, and let d be a real number such that $d > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{d+1}{z^d} \int_0^z t^{d-1} f(t)\, dt \tag{3.1}$$

also belongs to the class $P_\alpha[\beta,\gamma]$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \tag{3.2}$$

where

$$b_n = \left(\frac{d+1}{d+n}\right) a_n. \tag{3.3}$$

Therefore,

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)b_n = \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\left(\frac{d+1}{d+n}\right) a_n$$

$$\leq \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_n \leq 2\gamma(1-\beta), \tag{3.4}$$

since $f(z) \in P_\alpha[\beta,\gamma]$. Hence, by Lemma 1, $F(z) \in P_\alpha[\beta,\gamma]$.

THEOREM 5. Let the function $F(z) = z - \sum_{n=2}^{\infty} a_n z^n \; (a_n \geq 0)$ be in the class $P_\alpha[\beta,\gamma]$, and let d be a real number such that $d > -1$. Then the function f(z) defined by (3.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_n \left[\frac{(1+\gamma)C(\alpha,n)(d+1)}{2\gamma(1-\beta)(d+n)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{3.5}$$

The result is sharp.

PROOF. From (3.1), we have

$$f(z) = \frac{z^{1-d}\,(z^d F'(z))'}{(d+1)} \quad (d > -1)$$

$$= z - \sum_{n=2}^{\infty} \left(\frac{d+n}{d+1}\right) a_n\, z^n. \tag{3.6}$$

In order to obtain the required result it suffices to show that $|f'(z) - 1| < 1$ in $|z| < R^*$. Now

$$|f'(z) - 1| = \left|-\sum_{n=2}^{\infty} n\left(\frac{d+n}{d+1}\right) a_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} n\left(\frac{d+n}{d+1}\right) a_n |z|^{n-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{n=2}^{\infty} n\left[\frac{d+n}{d+1}\right]a_n \; |z|^{n-1} < 1. \tag{3.7}$$

But Lemma 1 confirms that

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} a_n \le 1. \tag{3.8}$$

Hence (3.7) will be satisfied if

$$\frac{n(d+n)}{(d+1)}|z|^{n-1} \le \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} \qquad (n \ge 2)$$

or if

$$|z| \le \left[\frac{(1+\gamma)C(\alpha,n)(d+1)}{2\gamma(1-\beta)(d+n)}\right]^{\frac{1}{n-1}} \qquad (n \ge 2). \tag{3.9}$$

The required result follows now from (3.9). The result is sharp for the function

$$f(z) = z - \frac{2\gamma(1-\beta)(d+n)}{n(1+\gamma)C(\alpha,n)(d+1)} z^n \qquad (n \ge 2). \tag{3.10}$$

4. Modified Hadamard Products

Let the functions $f_i(z)$ $(i = 1,2)$ be defined by (2.1).

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{n=2}^{\infty} a_{n,1}\, a_{n,2}\, z^n. \tag{4.1}$$

Theorem 6. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$, and $0 < \gamma \le 1$. Then $f_1 * f_2(z) \in P_\alpha[\delta(\alpha,\beta,\gamma),\gamma]$, where

$$\delta(\alpha,\beta,\gamma) = 1 - \frac{\gamma(1-\beta)^2}{2(1+\gamma)(1-\alpha)}. \tag{4.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [10], we need to find the largest $\delta = \delta(\alpha,\beta,\gamma)$ such that

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\delta)}\, a_{n,1}\, a_{n,2} \le 1. \tag{4.3}$$

Since

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\, a_{n,1} \le 1 \tag{4.4}$$

and

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} a_{n,2} \le 1, \tag{4.5}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} \sqrt{a_{n,1}\, a_{n,2}} \le 1. \tag{4.6}$$

Thus it is sufficient to show that

$$\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\delta)} a_{n,1}\, a_{n,2} \le \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} \sqrt{a_{n,1}\, a_{n,2}} \quad (n \ge 2), \tag{4.7}$$

that is, that

$$\sqrt{a_{n,1}\, a_{n,2}} \le \frac{(1-\delta)}{(1-\beta)}. \tag{4.8}$$

Note that

$$\sqrt{a_{n,1}\, a_{n,2}} \le \frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha,n)} \quad (n \ge 2). \tag{4.9}$$

Consequently, we need only to prove that

$$\frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha,n)} \le \frac{(1-\delta)}{(1-\beta)} \quad (n \ge 2), \tag{4.10}$$

or, equivalently, that

$$\delta \leq 1 - \frac{2\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \qquad (n \geq 2). \tag{4.11}$$

Since

$$A(n) = 1 - \frac{2\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \tag{4.12}$$

is an increasing function of n $(n \geq 2)$, for $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, letting $n = 2$ in (4.12), we obtain

$$\delta \leq A(2) = 1 - \frac{\gamma(1-\beta)^2}{2(1+\gamma)(1-\alpha)}, \tag{4.13}$$

which completes the proof of Theorem 6.

Finally, by taking the functions given by

$$f_i(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \quad (i = 1,2) \tag{4.14}$$

we can see that the result is sharp.

Theorem 7. Let the function $f_1(z)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, and the function $f_2(z)$ defined by (2.1) be in the class $P_\alpha[\tau,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \tau < 1$, and $0 < \gamma \leq 1$, then $f_1 *$

$f_2(z) \in P_\alpha[\zeta(\alpha,\beta,\tau,\gamma),\gamma]$, where

$$\zeta(\alpha,\beta,\tau,\gamma) = 1 - \frac{\gamma(1-\beta)(1-\tau)}{2(1+\gamma)(1-\alpha)}. \tag{4.15}$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 6, we get

$$\zeta \le B(n) = 1 - \frac{2\gamma(1-\beta)(1-\tau)}{n(1+\gamma)C(\alpha,n)} \quad (n \ge 2). \tag{4.16}$$

Since the function B(n) is an increasing function of n ($n \ge 2$), for $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$, $0 \le \tau < 1$, and $0 < \gamma \le 1$, letting n = 2 in (4.16) we obtain

$$\zeta \le B(2) = 1 - \frac{\gamma(1-\beta)(1-\tau)}{2(1+\gamma)(1-\alpha)}, \tag{4.17}$$

which evidently proves Theorem 7.

Finally the result is best possible for the functions

$$f_1(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \tag{4.18}$$

and

$$f_2(z) = z - \frac{\gamma(1-\tau)}{2(1+\gamma)(1-\alpha)} z^2. \tag{4.19}$$

Corollary 1. Let the functions $f_i(z)$ $(i = 1,2,3)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, then $f_1 * f_2 * f_3(z) \in P_\alpha[\eta(\alpha,\beta,\gamma),\gamma]$, where

$$\eta(\alpha,\beta,\gamma) = 1 - \frac{\gamma^2(1-\beta)^3}{4(1+\gamma)^2(1-\alpha)^2}. \tag{4.20}$$

The result is best possible for the functions

$$f_i(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \quad (i = 1,2,3). \tag{4.21}$$

Proof. From Theorem 6, we have $f_1 * f_2(z) \in P_\alpha[\delta(\alpha,\beta,\gamma),\gamma]$, where δ is given by (4.2). We use now Theorem 7, we get $f_1 * f_2 * f_3(z) \in P_\alpha[\eta(\alpha,\beta,\gamma),\gamma]$, where

$$\eta(\alpha,\beta,\gamma) = 1 - \frac{\gamma(1-\beta)(1-\delta)}{2(1+\gamma)(1-\alpha)} = 1 - \frac{\gamma^2(1-\beta)^3}{4(1+\gamma)^2(1-\alpha)^2}.$$

This completes the proof of Corollary 1.

Theorem 8. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$. Then the function

$$h(z) = z - \sum_{n=2}^{\infty} \left(a_{n,1}^2 + a_{n,2}^2\right) z^n \tag{4.21}$$

belongs to the class $P_{\alpha}[\varphi(\alpha,\beta,\gamma),\gamma]$, where

$$\varphi(\alpha,\beta,\gamma) = 1 - \frac{\gamma(1-\beta)^2}{(1+\gamma)(1-\alpha)}. \tag{4.22}$$

The result is sharp for the functions $f_i(z)$ $(i = 1,2)$ defined by (4.14).

Proof. By virtue of Lemma 1, we obtain

$$\sum_{n=2}^{\infty}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2 a_{n,1}^2 \le \left[\sum_{n=2}^{\infty}\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}a_{n,1}\right]^2 \le 1 \tag{4.23}$$

and

$$\sum_{n=2}^{\infty}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2 a_{n,2}^2 \le \left[\sum_{n=2}^{\infty}\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}a_{n,2}\right]^2 \le 1. \tag{4.24}$$

It follows from (4.23) and (4.24) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2\left(a_{n,1}^2 + a_{n,2}^2\right) \le 1. \tag{4.25}$$

Therefore, we need to find the largest $\varphi = \varphi(\alpha,\beta,\gamma)$ such that

$$\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\varphi)} \le \frac{1}{2}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2 \qquad (n \ge 2) \tag{4.26}$$

that is,

$$\varphi \leq 1 - \frac{4\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \qquad (n \geq 2). \tag{4.27}$$

Since

$$D(n) = 1 - \frac{4\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \tag{4.28}$$

is an increasing function of n $(n \geq 2)$, for $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, we readily have

$$\varphi \leq D(2) = 1 - \frac{\gamma(1-\beta)^2}{(1+\gamma)(1-\alpha)}, \tag{4.29}$$

and Theorem 8 follows at once.

5. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [14].

Definition 1. For real numbers $\eta > 0$, ρ and δ, the fractional integral operator $I_{0,z}^{\eta,\rho,\delta}$ is defined by

$$I_{0,z}^{\eta,\rho,\delta} f(z) = \frac{z^{-\eta-\rho}}{\Gamma(\eta)} \int_0^z (z-t)^{\eta-1} F(\eta+\rho, -\delta; \eta; 1-\frac{t}{z}) f(t)dt \tag{5.1}$$

where f(z) is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}), \qquad z \longrightarrow 0,$$

where

$$\varepsilon > \mathrm{Max}\ (0, \rho-\delta) - 1,$$

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{5.2}$$

where $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu+1)\ldots(\nu+n-1) & (n \in \mathbb{N} = \{1,2,\ldots\}), \end{cases} \tag{5.3}$$

and the multiplicity of $(z - t)^{\eta-1}$ is removed by requiring $\log (z-t)$ to be real when $z - t > 0$.

Remark. For $\rho = -\eta$, we note that

$$I_{0,z}^{\eta,-\eta,\delta} f(z) = D_z^{-\eta} f(z),$$

where $D_z^{-\eta} f(z)$ is the fractional integral of order η which was introduced by Owa ([4],[5]).

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [14].

Lemma 2. If $\eta > 0$ and $n > \rho - \delta - 1$, then

$$I_{o,z}^{\eta,\rho,\delta} z^n = \frac{\Gamma(n+1)\ \Gamma(n-\rho+\delta+1)}{\Gamma(n-\rho+1)\ \Gamma(n+\eta+\delta+1)} z^{n-\rho}. \tag{5.4}$$

With the aid of the Lemma 2, we prove

Theorem 9. Let $\eta > 0$, $\rho < 2$, $\eta+\delta > -2$, $\rho-\delta < 2$, $\rho(\eta+\delta) \leq 3\eta$. If the function f(z) defined by (1.12) is in the class $P_{\alpha}[\beta,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$. Then

$$|I_{o,z}^{\eta,\rho,\delta} f(z)| \geq \frac{\Gamma(2-\rho+\delta)\,|z|^{1-\rho}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)} \left\{1 - \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z|\right\} \tag{5.5}$$

and

$$|I_{o,z}^{\eta,\rho,\delta} f(z)| \leq \frac{\Gamma(2-\rho+\delta)\,|z|^{1-\rho}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)} \left\{1 + \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z|\right\} \tag{5.6}$$

for $z \in U_o$, where

$$U_o = \begin{cases} U & (\rho \leq 1) \\ U - \{0\} & (\rho > 1). \end{cases}$$

The equalities in (5.5) and (5.6) are attained for the function f(z) given by

$$f(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2. \tag{5.7}$$

Proof. By using Lemma 2, we have

$$I_{o,z}^{\eta,\rho,\delta} f(z) = \frac{\Gamma(2-\rho+\delta)}{\Gamma(2-\rho)\ \Gamma(2+\eta+\delta)} z^{1-\rho}$$

$$- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\ \Gamma(n-\rho+\delta+1)}{\Gamma(n-\rho+1)\ \Gamma(n+\eta+\delta+1)} a_n z^{n-\rho}. \tag{5.8}$$

Letting

$$H(z) = \frac{\Gamma(2-\rho)\ \Gamma(2+\eta+\delta)}{\Gamma(2-\rho+\delta)} z^{\rho} I_{o,z}^{\eta,\rho,\delta} f(z)$$

$$= z - \sum_{n=2}^{\infty} h(n)\ a_n z^n, \tag{5.9}$$

where

$$h(n) = \frac{(2-\rho+\delta)_{n-1} (1)_n}{(2-\rho)_{n-1} (2+\eta+\delta)_{n-1}} \qquad (n \geq 2), \tag{5.10}$$

we can see that h(n) is non-increasing for integers $n \geq 2$, and we have

$$0 < h(n) \le h(2) = \frac{2(2-\rho+\delta)}{(2-\rho)(2+\eta+\delta)}. \tag{5.11}$$

We note that $C(\alpha,n+1) \ge C(\alpha,n)$ for $0 \le \alpha \le \frac{1}{2}$ and $n \ge 2$. Hence by using Lemma 1, we obtain that

$$2(1+\gamma)C(\alpha,2)\sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_n \le 2\gamma(1-\alpha) \tag{5.12}$$

which implies that

$$\sum_{n=2}^{\infty} a_n \le \frac{\gamma(1-\alpha)}{2(1+\gamma)(1-\alpha)}. \tag{5.13}$$

Therefore, by using (5.11) and (5.13), we have

$$|H(z)| \ge |z| - h(2)\ |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\ge |z| - \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)}\ |z|^2 \tag{5.14}$$

and

$$|H(z)| \le |z| + h(2)\ |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\leq |z| + \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z|^2. \tag{5.15}$$

This completes the proof of Theorem 9.

REMARK 1. Taking $\rho = -\eta = -\lambda$ in Theorem 9, we get the result obtained by Owa and Ahuja [6, Theorem 10].

REMARK 2. Taking $\alpha = 1/2$ in the above results we obtain corresponding results for the class $P^*[\alpha,\gamma]$ defined by Gupta and Jain [2].

References

[1] O.P. Ahuja and H. Silverman, Convolutions of prestarlike functions, Internat. J. Math. Math. Sci. 6 (1983), 59-68.

[2] V.P. Gupta and P.K. Jain, Certain classes of univalent functions with negative coefficients. II, Bull. Austral. Math. Soc. 15 (1976), 467-473.

[3] T.H. MacGregor, The radius of convexity for starlike functions of order $\frac{1}{2}$, Proc. Amer. Math. Soc. 14 (1963), 71-76.

[4] S. Owa, On the distortion theorems. I, Kyungpook Math. J. 18 (1978), 53-59.

[5] S. Owa, Some applications of the fractional calculus,

Research Notes in Math. 138, Pitman, Boston, London and Melbourne, 1985, 164-175.

[6] S. Owa and O.P. Ahuja, A class of functions defined by using Hadamard product, Hokkaido Math. J. 15 (1986), 217-232.

[7] B. Pinchuk, On starlike and convex functions of order α, Duke Math. J. 35 (1968), 721-734.

[8] M.S. Robertson, On the theory of univalent functions, Ann. of Math. 37 (1936), 374-408.

[9] A. Schild, On starlike functions of order α, Amer. J. Math. 87 (1965), 65-70.

[10] A. Schild and H. Silverman, Convolutions of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 29 (1975), 99-107.

[11] T. Sheil-Small, H. Silverman and E.M. Silvia, Convolution multipliers and starlike functions, J. Analyse Math. 41 (1982), 181-192.

[12] H. Silverman and E. M. Silvia, Prestarlike functions with negative coefficients, Internat. J. Math. Math. Sci. 2 (1979), 427-439.

[13] H. Silverman and E. M. Silvia, Subclasses of

prestarlike functions, Math. Japon. 29 (1984), 929-935.

[14]. H. M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Appl. 131 (1988), 412-420.

S. Owa
Dept. of Math.
Faculty of Science and Tecnology
Kinki Univ.
Higoshi - Osaka, Osaka 577
Japan

H. M. Hossen and M. K. Aouf
Dept. of Math.
Faculty of Science
Univ. of Mansoura
Mansoura, Egypt